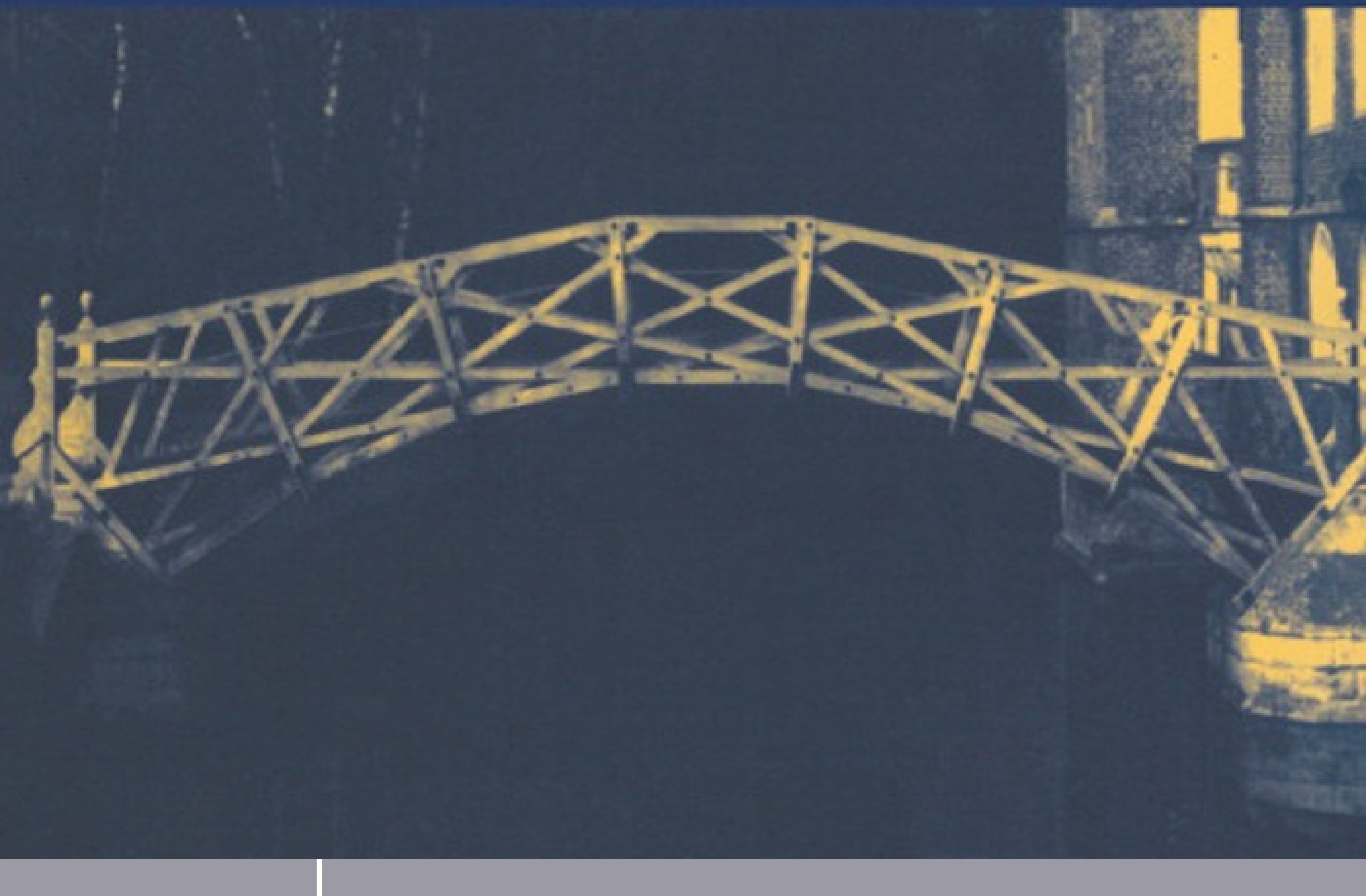
An Introduction to Mathematical Reasoning

numbers, sets and functions

PETER J. ECCLES



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I too will something make
    And joy in the making;
Altho' tomorrow it seem
Like the empty words of a dream
    Remembered on waking.

Robert Bridges, *Shorter poems.*

In loving memory of

Elizabeth Baron and John Baron

(Auntie Lizzie and Uncle Jack)

and those Woodside Bank summers

Contents

Preface

This book is based on lectures given at the Victoria University of Manchester to first year Honours Mathematics students including those taking joint or combined degrees. In common with most other British mathematics departments, the Manchester mathematics department has thoroughly reviewed its curriculum in recent years in the attempt to meet more adequately the needs of students who have experienced the effects of the great changes in the teaching of mathematics in schools, *as* well *as* the increased numbers of 'mature students' and students from non-standard backgrounds.

It was clear to us at the University of Manchester that we should completely rethink and broaden our curriculum, including material which we had previously expected students to know on entry to the course but also including introductory material on combinatorics, computer skills and numerical mathematics, *as* well *as* encouraging the development of problem solving skills.

A key ingredient of this new University of

Manchester curriculum is a module on Mathematical Reasoning whose purpose is to introduce the basic ideas of mathematical proof and to develop skills in writing mathematics, helping to bridge the gap between school and university mathematics. This book is based on this course module. The ability to write correct and clear mathematics is a skill which has to be acquired by observing experienced practitioners at work in lectures and tutorials, by learning to appreciate the details of mathematical exposition in books, and by a great deal of practice. It is a skill readily transferable to many other areas and its acquisition is likely to be one of the main benefits of a mathematics degree course for most students who may well make no further use of most of the specific mathematical content of the course!

There is no absolutely *correct* way of writing out a given proof. For example, it is necessary to take into account who the intended reader is. But, whoever the reader is, the proof should be expressed as clearly *as* possible and to achieve this the writer needs to understand the logic of the proof. Writing a proof is not separate from discovering the proof in the way that writing up a scientific experiment is separate from carrying out the experiment or performing a piece of music is separate from composing it. Attempts to write out a proof are an important part of the discovery process. Alison Leonard has written in a totally different context:

Not only are human ideas conveyed by language, they are actually formed by the language available to us.[†]

So it is in mathematics, where we find a parallel development of mathematical ideas and mathematical language.

This is reflected in this book. The emphasis is ‘on helping the reader to understand and construct proofs and to learn to write clear and concise mathematics. This can only be achieved by exploring some particular mathematical topics and the contexts chosen are set theory, combinatorics and number theory. These topics

- provide good examples to illustrate a range of basic methods of proof, in particular proof by induction and proof by contradiction,
- include some fundamental ideas which are part of the standard tool kit of any mathematician, such as functions and inverses, the binomial theorem, the Euclidean algorithm, the pigeonhole principle,

the fundamental theorem of arithmetic, and congruence,

- build on ideas met in early schooling illustrating ways in which familiar ideas can be formulated rigorously, for example counting or the greatest common divisor,
- include some of the all time great classic proofs, for example the Euclidean proofs of the irrationality of root 2 and the infinity of primes, but also ideas from throughout mathematical history so that there is an opportunity to present mathematics as a continually developing subject.

Roughly speaking, the first three parts of the book are about the basic language of mathematics and the final three parts are about number theory, illustrating how the ideas of the earlier parts are applied to some significant mathematics. The reader may find that the later parts contain some more straightforward material than the early parts simply because there is material on problem solving techniques which can then be practised on specific numerical examples. The topics selected for these later parts, the Euclidean algorithm, modular arithmetic and prime numbers, include material from the whole of mathematical history from classical Greek times to the present day.

I would encourage the reader in working through the first three parts not to expect to understand everything at first. Part I introduces various forms of mathematical statements and the standard methods of proof. Proof by contradiction and proof by induction are explained in detail and these methods are then used again and again throughout the remainder of the book: so many great theorems are proved by contradiction and some are included in this book. Part II covers the basic material on sets and functions which provides the language in which much mathematics is best expressed. It includes a leisurely discussion of universal and existential statements which are so important in university mathematics, particularly analysis or advanced calculus. This is not a book on mathematical logic but inevitably some ideas from the beginnings of that subject are included in these first two parts. The reader may find some of the material on counting in Part III to be more difficult than the rest of the book. It is best not to become discouraged by this but if necessary to move on to Part IV which is probably the easiest part but also includes some very striking and attractive results, returning to Part III when more familiarity with the language of sets and functions has been acquired.

This material on counting provides good practice in using the language of sets and functions in a very familiar context and also illustrates how a familiar process may be made mathematically precise and how this then enables the process to be extended to a less familiar context, counting infinite sets.

The book is divided into twenty four chapters and grew out of a series of twenty four lectures. However, there is far more material in most chapters than could reasonably be covered in a single lecture. A lecture course based on this book would need to be selective covering either a subset of the chapters or, more likely, a subset of the material in most or all of the chapters as the author’s lecture courses have done.

There is a great range of ability, experience and knowledge amongst students embarking on university mathematics courses and a real attempt has been made here to provide material meeting the needs of weaker or ill-prepared students whilst at the same time providing something which will interest and challenge the most able students. This is achieved by including material and, in particular, problems of varying difficulty; many of the over 250 problems are routine and computational but others are quite demanding. In the later stages of the book some significant developments of the ideas in the book are approached in the problems.

Each chapter concludes with a number of exercises many of which are very closely related to material in the chapter and intended to be relatively .straightforward and routine. Occasionally something a little harder is included. The reader is encouraged to work through all of these and full solutions for them are given at the back of the book. There are also six sets of problems, some similar to the exercises in order to consolidate the techniques involved but others more wide ranging and challenging. I hope that every reader will find some of the exercises and problems to be fun.

Mathematicians are not good at encouraging students to read around the subject but the intention here is that material in this book not covered in a lecture course will provide additional reading particularly for stronger students and, through the references given, lead to wider study in some areas. I have tried to write the book which I would have welcomed for my own students.

Finally, I am only too conscious that anyone who writes a book about how to write mathematics well lays themselves open to ridicule when the proofs in the book are found to be confused or inadequate. If the reader does find failings in the proofs in this book then I hope that acquiring the ability to see these failings will be seen as a useful step in developing that self-criticism which is necessary for the writing of clear and beautiful mathematics.

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Peter J. Eccles, Manchester, June 1997.

† See Alison Leonard, *Telling our stories: wrestling with a fresh language for the spiritual journey*, Dorton, Longman and Todd, 1995.

Part I

Mathematical statements and proofs

1

The language of mathematics

Pure mathematics is concerned with the exploration of mathematical concepts arising initially from the study of space and number. In order to capture and communicate mathematical ideas we must make statements about mathematical objects and much mathematical activity can be described as the formulation of mathematical statements and then the determination of whether or not such statements are true or false. It is important to be clear about what constitutes it mathematical statement and this is considered in this first chapter. We begin with simple statements and then examine ways of building up more complicated statements.

1.1 Mathematical statements

It is quite difficult to give a precise formulation of what a mathematical statement is and this will not be attempted in this book. The aim here is to enable the reader to recognize simple mathematical statements. First of all let us consider the idea of a *proposition*. A good working criterion is that *a proposition is a sentence which is either true* or *false (but not both)*. For the moment we are not so concerned about whether or not propositions are in fact true. Consider the following list.

(i) $1 + 1 = 2$.
(ii) $\pi = 3$.
(iii) 12 may be written as the sum of two prime numbers.
(iv) Every even integer greater than 2 may be written as the sum of two prime numbers.
(v) The square of every even integer is even.
(vi) n is a prime number.
(vii) $n^2 - 2n > 0$.
(viii) $m < n$.
(ix) $12 - 11$.
(x) π is a special number.

Of these the first five are propositions. This says nothing about whether or not they are true. In fact (i) is true and (ii) is false. Proposition (iii) is true since $12 = 7 + 5$ and 5 and 7 are prime numbers.† Propositions (iv) and (v) are general statements which cannot be proved by this sort of simple arithmetic. In fact it is easy to give a general proof of (v) (see Exercise 3.3) showing that it is true. However, at the time of writing, it is unknown whether (iv) is true or false; this statement is called the Goldbach conjecture after Christian Goldbach who suggested that it might be true in a letter to Leonhard Euler written in 1742.

On the other hand the others on the above list are not propositions. The next two, (vi) and (vii), become propositions once a numerical *value* is assigned to n. For example if $n = 2$ then (vi) is true and (vii) is false, whereas if $n = 3$ then they are both true. The next, (viii), becomes a proposition when values are assigned to both m and n, for example it is true for $m = 2$, $n = 3$ and false for $m = 3$, $n = 2$. Sentences of this type are called *predicates.* The symbols which need to be given values in order to obtain a proposition are called *free variables.*

The word *statement* will be used to denote either a proposition or a predicate. So in the above list the first eight items are statements. We will use a single capital letter P or Q to indicate a statement, or sometimes an expression like $P(m, n)$ to indicate a predicate, with the free variables listed in brackets.

The last two entries in the above list are not statements: (ix) is not even a sentence and (x) doesn't mean anything until we know what 'special' means. Very often mathematicians do give technical meanings to everyday words (as in 'prime' number used in (iii) and (iv) above and 'even' number used in (v)) and so if 'special' had been given such a meaning as a possible property of a number then (x) would be a proposition and so a statement. Of course the fact that (i) to (viii) are statements relies on a number of assumptions about the meanings of the symbols and words which have been taken for granted.

In particular it is quite complicated to give a precise definition of the number π in statement (ii). The number π was originally defined geometrically as the ratio between the length of the circumference of a circle and the length of the diameter of the circle. In order for this definition to be justified it is necessary to define the length of the circumference, a curved line, and having done that to prove that the ratio is independent of the size of the circle. This was achieved by the Greeks in classical times. A regular hexagon inscribed in a circle of unit diameter has circumference 3 and so it is clear that $\pi > 3$.[†] Archimedes showed by geometrical means in his book *On the measurement of the circle* that $3\frac{10}{71} < \pi < 3\frac{1}{7}$ and also provided a beautiful proof of the formula πr^2 for the area of a circle of radius r, The modern definition of π is usually as twice the least positive number for which the cosine function vanishes, and the details may be found in any text on analysis or advanced calculus.[‡]

1.2 Logical connectives

In mathematics we are often faced with deciding whether some given proposition is true or whether it is false. Many statements are really quite' complicated and are built up out of simpler statements using various 'logical connectives'. For the moment we restrict ourselves to the simplest of these: 'or', 'and' and 'not'. The truth or falsehood of a complicated statement is determined by the truth or falsehood of its component statements. It is important to be clear how this is done.

The connective 'or'

Suppose that we say

For integers a and b, $ab = 0$ if $a = 0$ or $b = 0$.

The statement '$a = 0$ or $b = 0$' is true if $a = 0$ (regardless of the value of b) and is also true if $b = 0$ (regardless of the value of a). Notice that the statement is true if both $a = 0$ and $b = 0$ are true. This is called the 'inclusive' use of 'or'. Its meaning is best made precise by means of a *truth table.* Any statement may be either true or false: we say that 'true' and 'false' are the two possible *truth values* for the statement. So given two statements P and Q each has two possible truth values giving four possible combinations in all. The truth table for 'or' which follows specifies the truth value for 'P or Q' corresponding to each possible combination of truth values for P and for Q, one line for each. In the table T indicates 'true' and F indicates 'false'.

Table 1.2.1

P	Q	P or Q
T	T	T
T	F	T
F	T	T
F	F	F

In this table, the truth values in the second line for example indicate that if P is true and Q is false then the statement 'P or Q' is true.

'P or Q' is called the *disjunction* of the two statements P and Q.

In everyday speech 'or' is often used in the exclusive sense as in the first sentence of this section: 'we are faced with deciding whether some given proposition is true or whether it is false', in which it is implicitly understood (and emphasized by the uses of the word 'whether') that it is not possible for a proposition to be both true and false.

To give another everyday example of the 'exclusive or', when we say 'everyone will travel there by bus or by train' this would normally be taken to mean that everyone uses one or other form of transport but not both. If we wanted to allow for someone using both we would probably say 'everyone will travel there by bus or by train or by both'. In practice the precise meaning of 'or' is made clear by the context or there is some ambiguity (but not both!). The meaning of mathematical statements must be precise and so we avoid these ambiguities by always using 'or' in the inclusive way determined by the above truth table.

The connective 'or' is sometimes hidden in other notation. For example when we write '$a \leqslant b$' where a and b are real numbers this is a shorthand for '$a < b$ or $a = b$'. Thus both the statements '$1 \leqslant 2$' and '$2 \leqslant 2$' are true.

Similarly '$a = \pm b$' is a shorthand for '$a = b$ or $a = -b$'. There is some possibility of confusion here for the true statement that $1 = \pm 1$ does not mean that 1 and ± 1 are interchangeable: 'if $x^2 = 1$ then $x = \pm 1$' is a true statement whereas 'if $x^2 = 1$ then $x = 1$' is a false statement! It is necessary to be clear that $a = \pm b$ is not asserting that a single equality is true but is asserting that one (or both) of two equalities $a = b$, $a \neq b$ is true.