Applied Mathematical Sciences
Volume 110

Applied Mathematical Sciences

1. *John:* Partial Differential Equations, 4th ed.
2. *Sirovich:* Techniques of Asymptotic Analysis.
3. *Hale:* Theory of Functional Differential Equations, 2nd ed.
4. *Percus:* Combinatorial Methods.
5. *von Mises/Friedrichs:* Fluid Dynamics.
6. *Freiberger/Grenander:* A Short Course in Computational Probability and Statistics.
7. *Pipkin:* Lectures on Viscoelasticity Theory.
8. *Giacoglia:* Perturbation Methods in Non-linear Systems.
9. *Friedrichs:* Spectral Theory of Operators in Hilbert Space.
10. *Stroud:* Numerical Quadrature and Solution of Ordinary Differential Equations.
11. *Wolovich:* Linear Multivariable Systems.
12. *Berkovitz:* Optimal Control Theory.
13. *Bluman/Cole:* Similarity Methods for Differential Equations.
14. *Yoshizawa:* Stability Theory and the Existence of Periodic Solution and Almost Periodic Solutions.
15. *Braun:* Differential Equations and Their Applications, 3rd ed.
16. *Lefschetz:* Applications of Algebraic Topology.
17. *Collatz/Wetterling:* Optimization Problems.
18. *Grenander:* Pattern Synthesis: Lectures in Pattern Theory, Vol. I.
19. *Marsden/McCracken:* Hopf Bifurcation and Its Applications.
20. *Driver:* Ordinary and Delay Differential Equations.
21. *Courant/Friedrichs:* Supersonic Flow and Shock Waves.
22. *Rouche/Habets/Laloy:* Stability Theory by Liapunov's Direct Method.
23. *Lamperti:* Stochastic Processes: A Survey of the Mathematical Theory.
24. *Grenander:* Pattern Analysis: Lectures in Pattern Theory, Vol. II.
25. *Davies:* Integral Transforms and Their Applications, 2nd ed.
26. *Kushner/Clark:* Stochastic Approximation Methods for Constrained and Unconstrained Systems.
27. *de Boor:* A Practical Guide to Splines.
28. *Keilson:* Markov Chain Models—Rarity and Exponentiality.
29. *de Veubeke:* A Course in Elasticity.
30. *Shiatycki:* Geometric Quantization and Quantum Mechanics.
31. *Reid:* Sturmian Theory for Ordinary Differential Equations.
32. *Meis/Markowitz:* Numerical Solution of Partial Differential Equations.
33. *Grenander:* Regular Structures: Lectures in Pattern Theory, Vol. III.
34. *Kevorkian/Cole:* Perturbation Methods in Applied Mathematics.
35. *Carr:* Applications of Centre Manifold Theory.
36. *Bengtsson/Ghil/Källén:* Dynamic Meteorology: Data Assimilation Methods.
37. *Saperstone:* Semidynamical Systems in Infinite Dimensional Spaces.
38. *Lichtenberg/Lieberman:* Regular and Chaotic Dynamics, 2nd ed.
39. *Piccini/Stampacchia/Vidossich:* Ordinary Differential Equations in $\mathbf{R}^n$.
40. *Naylor/Sell:* Linear Operator Theory in Engineering and Science.
41. *Sparrow:* The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors.
42. *Guckenheimer/Holmes:* Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields.
43. *Ockendon/Taylor:* Inviscid Fluid Flows.
44. *Pazy:* Semigroups of Linear Operators and Applications to Partial Differential Equations.
45. *Glashoff/Gustafson:* Linear Operations and Approximation: An Introduction to the Theoretical Analysis and Numerical Treatment of Semi-Infinite Programs.
46. *Wilcox:* Scattering Theory for Diffraction Gratings.
47. *Hale et al:* An Introduction to Infinite Dimensional Dynamical Systems—Geometric Theory.
48. *Murray:* Asymptotic Analysis.
49. *Ladyzhenskaya:* The Boundary-Value Problems of Mathematical Physics.
50. *Wilcox:* Sound Propagation in Stratified Fluids.
51. *Golubitsky/Schaeffer:* Bifurcation and Groups in Bifurcation Theory, Vol. I.
52. *Chipot:* Variational Inequalities and Flow in Porous Media.
53. *Majda:* Compressible Fluid Flow and System of Conservation Laws in Several Space Variables.
54. *Wasow:* Linear Turning Point Theory.
55. *Yosida:* Operational Calculus: A Theory of Hyperfunctions.
56. *Chang/Howes:* Nonlinear Singular Perturbation Phenomena: Theory and Applications.
57. *Reinhardt:* Analysis of Approximation Methods for Differential and Integral Equations.
58. *Dwoyer/Hussaini/Voigt (eds):* Theoretical Approaches to Turbulence.
59. *Sanders/Verhulst:* Averaging Methods in Nonlinear Dynamical Systems.
60. *Ghil/Childress:* Topics in Geophysical Dynamics: Atmospheric Dynamics, Dynamo Theory and Climate Dynamics.

(continued following index)

Odo Diekmann   Stephan A. van Gils
Sjoerd M. Verduyn Lunel   Hans-Otto Walther

Delay Equations

Functional-, Complex-, and Nonlinear Analysis

With 34 Illustrations

Springer-Verlag
New York Berlin Heidelberg London Paris
Tokyo Hong Kong Barcelona Budapest

Odo Diekmann
Centrum voor Wiskunde en Informatica
1090 GB Amsterdam
The Netherlands
and
Instituut voor Theoretische Biologie
Rijkuniversiteit Leiden
2311 GP Leiden
The Netherlands

Stephan A. van Gils
Faculteit der Toegepaste Wiskunde
Universiteit Twente
7500 AE Enschede
The Netherlands

Sjoerd M. Verduyn Lunel
Faculteit der Wiskunde en Informatica
Universiteit van Amsterdam
1018 TV Amsterdam
The Netherlands

Hans-Otto Walther
Mathematisches Institut
Justus-Liebig-Universität
35392 Giessen
Germany

Mathematics Subject Classification (1991): 76P05, 82C40, 82B40

Library of Congress Cataloging-in-Publication Data
Delay equations: functional-, complex-, and nonlinear analysis / Odo Diekmann . . . [et al.].
p. cm. – (Applied mathematical sciences; v. 110)
Includes bibliographical references and index.
ISBN-13:978-1-4612-8696-7 e-ISBN-13:978-1-4612-4206-2
DOI.10.1007/978-1-4612-4206-2

1. Delay differential equations. I. Diekmann, O. II. Series: Applied mathematical sciences (Springer-Verlag New York Inc.); v. 110.
QA1.A647 vol. 110
[QA371]
510 s–dc20
[515′.35] 94-41858

Printed on acid-free paper.

Production managed by Natalie Johnson; manufacturing supervised by Joseph Quatela.
Photocomposed using the authors' TEX files.

9 8 7 6 5 4 3 2 1

ISBN-13:978-1-4612-8696-7 Springer-Verlag New York Berlin Heidelberg

Preface

The aim of this book is to provide an introduction to the mathematical theory of infinite dimensional dynamical systems by focussing on a relatively simple, yet rich, class of examples, viz. those described by delay differential equations.

It is a textbook giving detailed proofs and many exercises, which is intended both for self-study and for courses at a graduate level. It should also be suitable as a reference for basic results. As the subtitle indicates, the book is about concepts, ideas, results and methods from linear functional analysis, complex function theory, the qualitative theory of dynamical systems and nonlinear analysis.

It gives a motivated introduction to the theory of semigroups of linear operators, emphasizing duality theory and neglecting analytic semigroups (thus it is complementary to an introduction to infinite dimensional dynamical systems focussing on the other relatively simple, yet rich, class of examples, i.e., scalar reaction diffusion equations in one space dimension). It contains an exposition of spectral theory, with special attention to those operators for which all spectral information is contained in an analytic matrix valued function. It introduces the calculus of exponential types of entire functions and exploits this calculus to investigate the behaviour of the resolvent of the generator at infinity, which is a main step to characterise the closure of the span of all eigenvectors and generalized eigenvectors and to investigate the (non-) existence of so-called "small" solutions, which converge to zero faster than any exponential. Essentially, these are Laplace transform methods.

The variation-of-constants formula is the main tool in the development of the local stability and bifurcation theory of equilibrium solutions of nonlinear problems. The center manifold and Hopf bifurcation are treated in detail. Stability of periodic solutions is discussed in terms of Floquet multipliers and Poincaré maps. Subsequently a more global point of view is adopted to study the existence of periodic solutions, in particular so-called slowly oscillating solutions. Here the topological degree and fixed-point theorems are the main tools. A survey of known results on the global dynamics of solutions of delay equations (including some results on chaotic behaviour) completes the book.

From the point of view of applications the most important chapter is perhaps the one on characteristic equations which deals, often by means of examples, with techniques to find the region in parameter space corresponding to the stability of a steady state. At the boundary of that region, bifurcations take place. A formula for the direction of Hopf bifurcation serves as an algorithm to compute this direction in concrete examples. This is often as far as one can get analytically to find out about the possibility of coexistence of local attractors.

After studying this book the reader should have a working knowledge of applied functional analysis and dynamical systems. For purely minded analysts we expect that they become aware of the charm of concrete problems, where often the main difficulty is to find the right mathematical setting. For application oriented readers we expect that they learn to appreciate the extra understanding that mathematical rigour often entails. For people trained in ordinary differential equations the book shows what aspects of operator theory are essential when working in infinite dimensional state spaces. For readers with an operator background it introduces the main ideas concerning the behaviour of dynamical systems. Thus we hope that many different types of readers will find something of value in the book and will, while reading, experience some of the same enjoyment that we had while writing.

It is NOT a handbook for the use of delay equations as mathematical models of physical or biological phenomena. (In fact our opinion is that it is dangerous to model directly in terms of delay equations: careful modelling requires a mechanistic interpretation of the state of a system; of course it is perfectly all right if a delay equation results in the end, possibly after some transformation [255], but one should avoid starting to think in terms of such equations.) Throughout the book, however, it is shown (most of the time by means of exercises) how age dependent population models are covered by exactly the same mathematical theory.

The book was written in many episodes, scattered over a period of approximately six years. Often the obligation to restart to work on it felt like a burden, but when a little later other duties forced us to stop working on it, this felt as an even bigger nuisance. In between, fortunately, it was a pleasure. So now that the project is finished we feel mostly relief but in addition a little bit of excitement since now, finally, the fruits of our efforts are ready for the ultimate test of any book: do you, reader, like it or not?

Amsterdam, December 1994

Odo Diekmann
Stephan van Gils
Sjoerd Verduyn Lunel
Hans-Otto Walther

Contents

Preface v

0 Introduction and preview 1
0.1 An example of a retarded functional differential equation 1
0.2 Solution operators 3
0.3 Synopsis 5
0.4 A few remarks on history 10

I Linear autonomous RFDE 11
I.1 Prelude: a motivated introduction to functions of bounded variation 11
I.2 Linear autonomous RFDE and renewal equations 15
I.3 Solving renewal equations by Laplace transformation 23
I.4 Estimates for $\det \Delta(z)$ and related quantities 28
I.5 Asymptotic behaviour for $t \to \infty$ 31
I.6 Comments 35

II The shift semigroup 36
II.1 Introduction 36
II.2 The prototype problem 37
II.3 The dual space 40
II.4 The adjoint shift semigroup 41
II.5 The adjoint generator and the sun subspace 42
II.6 The prototype system 49
II.7 Comments 50

III Linear RFDE as bounded perturbations 51
III.1 The basic idea, followed by a digression on weak* integration 51
III.2 Bounded perturbations in the sun-reflexive case 54
III.3 Perturbations with finite dimensional range 64
III.4 Back to RFDE 69

III.5 Interpretation of the adjoint semigroup 76
III.6 Equivalent description of the dynamics 78
III.7 Complexification 80
III.8 Remarks about the non-sun-reflexive case 89
III.9 Comments 94

IV Spectral theory 95
IV.1 Introduction 95
IV.2 Spectral decomposition for eventually compact semigroups 96
IV.3 Delay equations 104
IV.4 Characteristic matrices, equivalence and Jordan chains 109
IV.5 The semigroup action on spectral subspaces for delay equations 123
IV.6 Comments 134

V Completeness or small solutions? 135
V.1 Introduction 135
V.2 Exponential type calculus 137
V.3 Completeness 140
V.4 Small solutions 150
V.5 Precise estimates for $|\Delta(z)^{-1}|$ 158
V.6 Series expansions 166
V.7 Lower bounds and the Newton polygon 170
V.8 Noncompleteness, series expansions and examples 178
V.9 Arbitrary kernels of bounded variation 184
V.10 Comments 191

VI Inhomogeneous linear systems 193
VI.1 Introduction 193
VI.2 Decomposition in the variation-of-constants formula 194
VI.3 Forcing with finite dimensional range 196
VI.4 RFDE 197
VI.5 Comments 200

VII Semiflows for nonlinear systems 201
VII.1 Introduction 201
VII.2 Semiflows 202
VII.3 Solutions to abstract integral equations 208
VII.4 Smoothness 214
VII.5 Linearization at a stationary point 222
VII.6 Autonomous RFDE 231
VII.7 Comments 241

VIII Behaviour near a hyperbolic equilibrium ... 242
VIII.1 Introduction ... 242
VIII.2 Spectral decomposition ... 243
VIII.3 Bounded solutions of the inhomogeneous linear equation ... 246
VIII.4 The unstable manifold ... 247
VIII.5 Invariant wedges and instability ... 253
VIII.6 The stable manifold ... 257
VIII.7 Comments ... 258

IX The center manifold ... 259
IX.1 Introduction ... 259
IX.2 Spectral decomposition ... 260
IX.3 Bounded solutions of the inhomogeneous linear equation ... 262
IX.4 Modification of the nonlinearity ... 264
IX.5 A Lipschitz center manifold ... 265
IX.6 Contractions on embedded Banach spaces ... 267
IX.7 The center manifold is of class C^k ... 271
IX.8 Dynamics on and near the center manifold ... 276
IX.9 Parameter dependence ... 277
IX.10 A double eigenvalue at zero ... 280
IX.11 Comments ... 286

X Hopf bifurcation ... 287
X.1 Introduction ... 287
X.2 The Hopf bifurcation theorem ... 287
X.3 The direction of bifurcation ... 292
X.4 Comments ... 301

XI Characteristic equations ... 302
XI.1 Introduction: an impressionistic sketch ... 302
XI.2 The region of stability in a parameter plane ... 305
XI.3 Strips ... 312
XI.4 Case studies ... 317
XI.5 Comments ... 338

XII Time-dependent linear systems ... 339
XII.1 Introduction ... 339
XII.2 Evolutionary systems ... 340
XII.3 Time-dependent linear RFDE ... 343
XII.4 Invariance of $X^{\odot}$: a counterexample and a sufficient condition ... 345
XII.5 Perturbations with finite dimensional range ... 348
XII.6 Comments ... 354

XIII Floquet Theory 355
XIII.1 Introduction 355
XIII.2 Preliminaries on periodicity and a stability result 356
XIII.3 Floquet multipliers 358
XIII.4 Floquet representation on eigenspaces 360
XIII.5 Comments 363

XIV Periodic orbits 364
XIV.1 Introduction 364
XIV.2 The Floquet multipliers of a periodic orbit 365
XIV.3 Poincaré maps 368
XIV.4 Poincaré maps and Floquet multipliers 372
XIV.5 Comments 376

XV The prototype equation for delayed negative feedback: periodic solutions 379
XV.1 Delayed feedback 379
XV.2 Smoothness and oscillation of solutions 382
XV.3 Slowly oscillating solutions 385
XV.4 The a priori estimate for unstable behaviour 391
XV.5 Slowly oscillating solutions which grow away from zero, periodic solutions 399
XV.6 Estimates, proof of Theorem 5.5(i) and (iii) 408
XV.7 The fixed-point index for retracts in Banach spaces, Whyburn's lemma 412
XV.8 Proof of Theorem 5.5(ii) and (iv) 415
XV.9 Comments 421

XVI On the global dynamics of nonlinear autonomous differential delay equations 426
XVI.1 Negative feedback 426
XVI.2 A limiting case 430
XVI.3 Chaotic dynamics in case of negative feedback 439
XVI.4 Mixed feedback 441
XVI.5 Some global results for general autonomous RFDE 442

Appendices

I Bounded variation, measure and integration 443
I.1 Functions of bounded variation 443
I.2 Abstract integration 446

II Introduction to the theory of strongly continuous semigroups of bounded linear operators and their adjoints 452

II.1 Strongly continuous semigroups 452
II.2 Interlude: absolute continuity 458
II.3 Adjoint semigroups 460
II.4 Spectral theory and asymptotic behaviour 468

III The operational calculus 474

III.1 Vector-valued functions 474
III.2 Bounded operators 476
III.3 Unbounded operators 477

IV Smoothness of the substitution operator 482

V Tangent vectors, Banach manifolds and transversality 492

V.1 Tangent vectors of subsets of Banach spaces 492
V.2 Banach manifolds 492
V.3 Submanifolds and transversality 494

VI Fixed points of parameterized contractions 497

VII Linear age-dependent population growth: elaboration of some of the exercises 500

VIII The Hopf bifurcation theorem 505

References 514

Index 530

List of symbols 533

List of notation 533

Chapter 0

Introduction and preview

0.1 An example of a retarded functional differential equation

Imagine a biological population composed of adult and juvenile individuals. Let $N(t)$ denote the density of adults at time t. Assume that the length of the juvenile period is exactly h units of time for each individual. Assume that adults produce offspring at a per capita rate α and that their probability per unit of time of dying is μ. Assume that a newborn survives the juvenile period with probability ρ and put $r = \alpha\rho$. Then the dynamics of N can be described by the differential equation

$$\dot{N}(t) = -\mu N(t) + rN(t-h) \tag{1.1}$$

which involves a nonlocal term, where N has argument $t - h$, since newborns become adults with some delay. So the rate of change of N involves the current as well as the past values of N. Such equations are called Retarded Functional Differential Equations (RFDE) or, alternatively, Delay Equations.

Equation (1.1) describes the change in N. To fix N, we need an initial condition, say at $t = 0$ (i.e., we start our clock at the time we prescribe the condition).

Example 1.1. The solutions $t \mapsto \sin(\frac{\pi}{2}(t+\frac{1}{2}))$ and $t \mapsto \cos(\frac{\pi}{2}(t+\frac{1}{2}))$ of the equation

$$\dot{x}(t) = -\frac{\pi}{2}x(t-1)$$

coincide at $t = 0$.

It is not enough to specify $N(0)$, since we need to know what to take instead of $rN(t-h)$ for $0 \le t < h$. So we have to prescribe a function on an interval of length h. The most convenient (though not the most natural from a biological point of view) manner to do this is to prescribe N on the interval $[-h, 0]$ and then to use (1.1) for $t \ge 0$.

So we supplement (1.1) by

$$N(\theta) = \varphi(\theta), \qquad -h \le \theta \le 0, \tag{1.2}$$

where φ is a given function. Explicitly we then have for $t \in [0, h]$

$$N(t) = \varphi(0)e^{-\mu t} + r \int_0^t e^{-\mu(t-\tau)}\varphi(\tau - h)\, d\tau. \tag{1.3}$$

Using this expression we can give an expression for N on the interval $[h, 2h]$. Etcetera. Thus the *method of steps* and elementary theory of ordinary differential equations (ODE) provide us with an existence and uniqueness proof.

The key question, of course, is whether the population will ultimately grow without bound or become extinct. In other words, we want to determine the asymptotic behaviour for $t \to \infty$ and how this depends on the parameters r and μ. If we formally substitute $N(t) = N(0)e^{zt}$ into (1.1), we arrive at the *characteristic equation*

$$z = -\mu + re^{-zh}. \tag{1.4}$$

Exercise 1.2.

(i) Show that (1.4) has exactly one real root. Call this root λ_d.

(ii) Show that $\operatorname{Re}\lambda < \lambda_d$ for all other roots λ (the subscript d refers to "dominant" and this qualification should now be clear).

(iii) Show that $\lambda_d > 0$ if $r\mu^{-1} > 1$, whereas $\lambda_d < 0$ if $r\mu^{-1} < 1$.

(iv) Verify that $r\mu^{-1}$ can be interpreted as the expected number of offspring produced by a newborn individual and that, consequently, the result of (iii) is exactly what one expects on the basis of the biological interpretation.

Combining the results of this exercise with our intuition (derived, say, from the theory of ODE) we are led to

Conjecture 1.3. If $\varphi(\theta) \ge 0$ with φ not identically zero, then

$$N(t) \sim e^{\lambda_d t} \quad \text{for } t \to \infty$$

and so the population will grow exponentially when $r > \mu$ and become extinct when $r < \mu$.

In this book we shall introduce techniques and prove general theorems from which the correctness of this conjecture follows. In Chapter I we shall use Laplace transform methods to study *linear* equations, like (1.1), and find that they are quite sufficient for this class of equations. However, if we go beyond and study *nonlinear* equations we need a different perspective as well as other methods.

If competition during the juvenile period influences the probability ρ of survival, we have to replace (1.1) by something else. The equation

$$\dot{N}(t) = -\mu N(t) + f(N(t-h)) \tag{1.5}$$

describes the situation in which competition takes place among individuals in the same age group only. Note that the appropriate initial condition again takes the form (1.2). What we now want is a qualitative theory for equations like (1.5) in much the same spirit as the one for ODE. In this book we shall develop the basic elements of such a theory of dynamical systems in infinite dimensional spaces, using delay equations like (1.1) and (1.5) and similar age-structured population models as our motivating examples.

0.2 Solution operators

In the preceding section we noticed that the information contained in a function φ defined on $[-h, 0]$ is needed and suffices to uniquely fix the future. Hence such a function qualifies as the *state*, at time 0, of the system we describe. Let, for an autonomous (i.e., time translation invariant) system, $S(t)$ denote the operator assigning to the state at some time the state t units of time later. Then necessarily, because of this interpretation,

(i) $S(0) = I$ (the identity operator),

(ii) $S(t+s) = S(t)S(s), \qquad t, s \geq 0,$

where the second property derives from the *uniqueness*. These properties are summarized by saying that the family $\{S(t)\}_{t\geq 0}$ forms a one-parameter semigroup of operators (the adjective "semi" expresses that backward solutions do not necessarily exist or, in other words, that t is restricted to nonnegative values). For a given initial state φ the *orbit* through φ is the subset $\{S(t)\varphi \mid t \geq 0\}$ of the state space. Ever since Poincaré, an important aim in the theory of dynamical systems is to give a qualitative, geometric, description of the collection of all orbits, the so-called phase portrait, possibly restricting attention to the neighbourhood of some steady state (an orbit consisting of one point), periodic solution (a closed orbit) or more complicated invariant set. In bifurcation theory one studies changes in the qualitative properties as parameters involved in the definition of $\{S(t)\}_{t\geq 0}$ vary.

If we want to let these ideas bear on delay equations, we have to specify the state space and to elaborate the definition of $S(t)$.

Exercise 2.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $h > 0$. Use the method of steps to prove that each continuous initial function $\varphi : [-h, 0] \to \mathbb{R}$ extends to a continuous function $N : [-h, \infty) \to \mathbb{R}$ which is differentiable for $t > 0$ and such that (1.2) and, for $t > 0$, (1.5) are satisfied. Show that N is uniquely determined by φ.

This exercise suggests choosing as a state space the Banach space

$$X = C\big([-h,0],\mathbb{R}\big)$$

of continuous real functions on the interval of length h, provided with the supremum norm. This turns out to be a good choice. Note, however, that also data from larger spaces of measurable and bounded functions on the interval of length h would uniquely define solutions (in a slightly weaker sense) on the domain $[-h,\infty)$, provided we specify precisely the value at $t=0$. In Chapter II we will find that such a space enters the scene quite naturally even if we start doing analysis in X as defined above.

The state at time t is the "piece" of the extended function N on the interval of length h preceding t, i.e., $\theta \mapsto N(t+\theta;\varphi)$, for $-h \le \theta \le 0$.

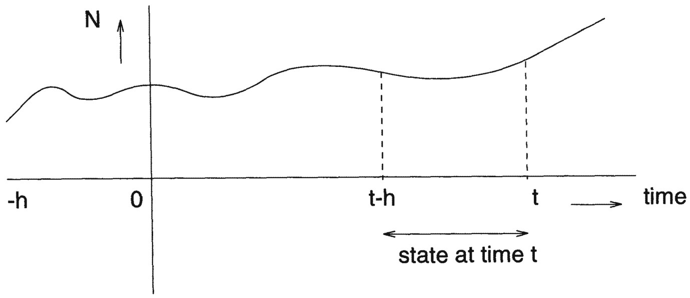

Fig. 0.1. The state at time t.

So we restrict $N(\,\cdot\,;\varphi)$ to the interval $[t-h,t]$ and then, in order to have again an element of X, we perform a shift back to the interval $[-h,0]$. In still other words and symbols,

$$S(t)\varphi = N_t(\,\cdot\,;\varphi),$$

where we use the notation $x_t(\theta) = x(t+\theta)$. We emphasize that this definition involves two ingredients: we extend φ by solving the RFDE and then we select the relevant piece of information by a shift along the extended function.

Exercise 2.2.

(i) Verify that we obtain, in this manner, a semigroup of operators, starting from equation (1.5).

(ii) Verify that the semigroup $S(t)$ is strongly continuous, which means by definition that for any $\varphi \in X$,

$$\|S(t)\varphi - \varphi\| \to 0 \quad \text{as } t \downarrow 0.$$

Exploit the semigroup property to conclude from this that $t \mapsto S(t)\varphi$ is continuous, i.e., orbits are continuous curves.

The *generator* (in this book denoted by A) of a semigroup of operators is by definition the derivative at $t = 0$. For an infinite dimensional state space this is, as a rule, an unbounded operator. In the theory of partial differential equations one often starts from an abstract differential equation

$$\frac{du}{dt} = Au$$

and then shows that there is a semigroup of solution operators associated with A. Since delay equations are not exactly of this form, the theory for these proceeds somewhat differently. In particular, one first defines the semigroup constructively and only then "computes" the generator.

0.3 Synopsis

In this book we shall study the qualitative properties of the solution operators of delay equations. The emphasis is on autonomous equations, which describe systems for which there are no time-varying inputs. The delays in the equations will always be bounded. See Hino, Murakami and Naito [125] for the theory of equations with unbounded delay.

In the first six chapters we restrict attention to linear equations.

Chapter I deals with the Laplace transform method, a classical approach to represent and analyse solutions of linear autonomous RFDE. First, the one-to-one relation between RFDE and renewal equations (RE) is explained. The latter are Volterra integral equations of convolution type. The convolution character makes them suitable for (inverse) Laplace techniques.

An asymptotic expansion of the solution of the RE is obtained by shifting the path of integration of the inverse Laplace transform to the left. The solutions of the characteristic equation of the RFDE are encountered as poles in this process. Crossing of a finite number of poles means "splitting off" an exponential polynomial (sums of exponentials with polynomial coefficients).

So by this procedure, the solution of an initial-value problem for a RFDE is represented by an exponential polynomial and a remainder term. Several questions arise. What can be said about the convergence of the asymptotic series that is obtained by including more and more poles? If there is no convergence for all initial data, can one determine data for which one has convergence? What can be said about the remaining solutions?

In Chapters II and III, the concept of a solution operator is introduced. As already remarked in Section 0.2, there are two ingredients involved in its construction: extending the function and shifting. In Chapter II the role of the shifting part is analysed in great detail for a linear equation where the extension part is trivial, i.e., the right hand side of the equation is zero. This defines a linear semigroup $\{T_0(t)\}$ on the space X. We compute the adjoint semigroup, $\{T_0^*(t)\}$, on X^*, and its restriction to the largest domain of strong continuity. This is denoted by $\{T_0^{\odot}(t)\}$ ($\odot$ is pronounced as sun) acting on $X^{\odot}$. With assiduity we go through the process of taking the adjoint and restricting to the domain of strong continuity once more, to obtain $\{T_0^{\odot *}(t)\}$ on $X^{\odot *}$ and $\{T_0^{\odot\odot}(t)\}$ on $X^{\odot\odot}$. In the case of delay equations $X^{\odot\odot}$ is isomorphic to X and we call the semigroup $\odot$-reflexive.

It so happens that the space $X^{\odot *}$ is essential for the precise functional analytic formulation of the variation-of-constants formula,which is the most important vehicle for the transfer of results from linear to nonlinear differential equations in local stability and bifurcation theory. It involves, in the case of delay equations, objects which do not “live” in the space X, but in the larger space $X^{\odot *}$. This has made the theory of delay equations somewhat mysterious for people with a functional analytic background. This difficulty was overcome when sun-star calculus was developed in an attempt to unify the theories of delay equations and age-structured population dynamics. One of the reasons to write this book was to demonstrate how sun-star calculus can be used with great advantage when developing the basic theory of RFDE.

In Chapter III it is shown that the general linear autonomous RFDE, with semigroup $\{T(t)\}$, can be viewed as a bounded perturbation of the above-mentioned trivial RFDE. For a bounded perturbation B, mapping X into $X^{\odot *}$, the variation-of-constants equation

$$u(t) = T_0(t-s)u(s) + \int_s^t T_0^{\odot *}(t-\tau)Bu(\tau)\,d\tau, \tag{3.1}$$

which is an abstract integral equation (AIE) involving the notion of a weak* integral, is derived. The linear autonomous RFDE is an example of a bounded perturbation with finite dimensional range and in this case, reduction of the abstract integral equation to a (finite dimensional) renewal equation is possible. Thus we see the connection between the abstract approach and the direct approach of Chapter I.

Chapter IV deals first with the spectral decomposition of the state space for an eventually compact semigroup, such as the one associated with a RFDE. Next it is shown how one can obtain all spectral information from the characteristic matrix (here we need the theory of Jordan chains and the notion of equivalence of operator valued functions). In particular we recover within the abstract framework the decomposition of solutions into an exponential polynomial and a lower order, for $t \to \infty$, remainder as found in Chapter I by the Laplace transform method.

Chapter V is concerned with answers to questions which arose in Chapter I. Results about entire functions, growth properties and exponential type calculus prepare the way to take up the question of convergence of the asymptotic expansions for solutions to linear autonomous RFDE and associated RE. Sufficient conditions for uniform convergence on unbounded intervals are given. The fact that one needs conditions to ensure convergence reflects the possible existence of *small solutions*, i.e., solutions which decay faster than exponentially as $t \to \infty$. These solutions cannot be approximated by the expansions of Chapter I.

The nonexistence of small solutions is characterized in terms of growth properties of the characteristic function whose zeros determine the spectrum of the generator. A closely related problem is: How far is the system of generalized eigenvectors from being complete (in the sense that the closure of the span is the full state space X)? A characterization of completeness and a description of the general situation are given.

Chapter VI on inhomogeneous equations is a link between linear and nonlinear theory. Following the philosophy of Chapter III we are led to consider the AIE

$$(3.2) \qquad u(t) = T(t-s)u(s) + \int_s^t T^{\odot *}(t-\tau)F(\tau)\, d\tau, \qquad t \geq s,$$

with F a continuous mapping into $X^{\odot *}$. This is the integrated version of the abstract differential equation

$$(3.3) \qquad \frac{du}{dt}(t) = A^{\odot *}u(t) + F(t).$$

Solutions u given by (3.2) are usually called mild solutions of (3.3). In a general context the question in what sense u given by (3.2) satisfies (3.3) is a delicate one. It turns out that this is irrelevant in the context of delay equations. A one-to-one relation between solutions of the inhomogeneous linear RFDE and the corresponding AIE is given explicitly. As an application we derive the Fredholm alternative for periodic inhomogeneities F.

In Chapter VII we collect basic results on existence, uniqueness and smoothness of solutions of nonlinear equations in a parameter dependent context. We derive the principle of linearized stability, which asserts that the stability of a stationary point can be inferred by examining the stability with respect to the linearized equation.

The next three chapters deal with the local theory for nonlinear RFDE. Linearization of a nonlinear RFDE at a stationary point yields the strongly continuous semigroup of solution operators $T(t)$. The spectrum of the generator A is decomposed into the following parts: σ_- in the open left half-plane, σ_0 on the imaginary axis and σ_+ in the open right half-plane. The spectral sets $\sigma_-, \sigma_0, \sigma_+$ define subspaces X_-, X_0, X_+, respectively, which are positively invariant under the semigroup; trajectories in the stable space X_- converge exponentially to 0 as $t \to \infty$. Trajectories in the center space

X_0 and in the unstable space X_+ have extensions in these spaces which are defined on the whole real line. The extensions in X_+ converge to 0 as $t \to -\infty$.

In Chapter VIII we obtain under the hyperbolicity assumption

$$\sigma_0 = \emptyset$$

the existence of the local stable and unstable manifolds. These are tangent to X_- and X_+, respectively, and consist of orbits which behave qualitatively like the orbits of $\{T(t)\}$ in X_- and in X_+, respectively.

Chapter IX contains the construction of local center manifolds, which are tangent to X_0 and positively invariant. Included is a detailed proof that these manifolds are continuously differentiable in case the nonlinearity has this property.

Chapter X applies the previous theory to the phenomenon of Hopf bifurcation, i.e., the appearance of periodic orbits close to a stationary point of a one-parameter family of equations. We derive a formula for the direction of bifurcation which determines whether the bifurcating periodic orbits appear below or above the critical parameter, which is, in many important situations, decisive for their stability character.

The local results of Chapters VIII, IX and X rely on hypotheses about the spectrum or, in other words, about the roots of the characteristic equation associated with the linearized RFDE. There is no general theory which describes how to locate these roots. In Chapter XI we collect a number of techniques, often ad hoc and illustrated in the context of a concrete example, which help to determine the position of the roots relative to the imaginary axis. For most of the examples we also compute the direction of Hopf bifurcation. These are probably the most useful parts of the book for applied mathematicians and scientists that seek to obtain conclusions for concrete models taking the form of a RFDE or something similar.

When we linearize about an orbit that consists of more than one point, we obtain a nonautonomous linear problem. Such problems are touched upon briefly in Chapter XII where we discuss the evolutionary system of linear solution operators

$$U(t,s), \quad t \geq s,$$

which is associated with the initial-value problem for a time-dependent linear RFDE. Then in Chapter XIII we concentrate on the periodic case and the definition of Floquet multipliers. In the ODE context, Floquet theory is used to transform a periodic linear system to an autonomous linear system. We derive an analogous result in the abstract setting.

The results of Chapter XIII are used in Chapter XIV where we consider nonlinear (autonomous) AIE and RFDE in neighbourhoods of periodic orbits, i.e., close to the simplest invariant sets which are not points. The behaviour of orbits close to a periodic orbit is described in terms of the Poincaré map which assigns to points in a transversal hyperplane the intersection of the corresponding trajectory with the same hyperplane at a later time. The Poincaré map has a fixed point at the intersection of the periodic orbit with the hyperplane.

The construction of the Poincaré map requires local smoothness of the semiflow. In the case of RFDE this will, in general, only be satisfied if the period considered is larger than the delay h in the equation.

The linearization of a Poincaré map at its fixed point on the periodic orbit is closely related to the monodromy operators associated with the linear variational equation along the periodic orbit. We establish the precise relations between these maps, their spectra, their generalized eigenspaces and Floquet multipliers under the condition that all spectral points of the monodromy operators except 0 are isolated.

In the final two chapters we no longer concentrate on the neighbourhood of a special solution but instead adopt a more global point of view. As a consequence we need methods from nonlinear analysis beyond the implicit function theorem, in particular the fixed-point index and global bifurcation theorems.

In Chapter XV we study existence of periodic solutions for the nonlinear autonomous RFDE

$$\dot{x}(t) = f(x(t-h)) \tag{3.4}$$

from a global point of view, for nonlinearities which model negative feedback with respect to a stationary state. We consider continuous functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$f(0) = 0 \quad \text{and} \quad \xi f(\xi) < 0 \quad \text{for all } \xi \neq 0.$$

An example for which periodic solutions can explicitly be computed is given.

Scaling the variable t and setting $\alpha = h$ we obtain a one-parameter family of equations

$$\dot{x}(t) = \alpha f(x(t-1)), \quad \alpha > 0, \tag{3.5}$$

on the state space $X = C\big([-1,0],\mathbb{R}\big)$. We prove a global bifurcation theorem due to Nussbaum which asserts, under minimal smoothness and boundedness conditions on f, that an unbounded continuum of nontrivial periodic orbits bifurcates from the zero solution at the critical parameter value

$$\alpha_0 = \frac{\pi}{2f'(0)}.$$

For every $\alpha > \alpha_0$ there exist periodic solutions. The periodic orbits are obtained from fixed points of a return map which is defined by the intersection of trajectories with a cone, analogous to a Poincaré map. We employ a deeper analysis of invariant sets and unstable behaviour of trajectories which in turn permits a straightforward calculation of the index. The methods we introduce play a role also in further studies of the global dynamics generated by nonlinear RFDE.