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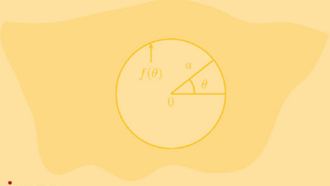
$$\theta) = \frac{(r^2 - a^2)}{2\pi} \int_{-\pi}^{\pi} \frac{f(\phi)}{a^2 + r^2 - 2ra\cos(\theta - \phi)} d\phi, \quad r > a.$$

Ravi P. Agarwal
Donal O'Regan

Ordinary and Partial Differential Equations

With Special Functions, Fourier Series, and Boundary Value Problems



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Ordinary and Partial Differential Equations

With Special Functions, Fourier Series, and Boundary Value Problems

Springer

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Dedicated to our Sons

Hans Agarwal and Daniel Joseph O'Regan

Preface

This book comprises 50 class-tested lectures which both the authors have given to engineering and mathematics major students under the titles *Boundary Value Problems* and *Methods of Mathematical Physics* at various institutions all over the globe over a period of almost 35 years. The main topics covered in these lectures are power series solutions, special functions, boundary value problems for ordinary differential equations, Sturm–Liouville problems, regular and singular perturbation techniques, Fourier series expansion, partial differential equations, Fourier series solutions to initial-boundary value problems, and Fourier and Laplace transform techniques. The prerequisite for this book is calculus, so it can be used for a senior undergraduate course. It should also be suitable for a beginning graduate course because, in undergraduate courses, students do not have any exposure to various intricate concepts, perhaps due to an inadequate level of mathematical sophistication. The content in a particular lecture, together with the problems therein, provides fairly adequate coverage of the topic under study. These lectures have been delivered in one year courses and provide flexibility in the choice of material for a particular one-semester course. Throughout this book, the mathematical concepts have been explained very carefully in the simplest possible terms, and illustrated by a number of complete workout examples. Like any other mathematical book, it does contain some theorems and their proofs.

A detailed description of the topics covered in this book is as follows: In Lecture 1 we find explicit solutions of the first-order linear differential equations with variable coefficients, second-order homogeneous differential equations with constant coefficients, and second-order Cauchy–Euler differential equations. In Lecture 2 we show that if one solution of the homogeneous second-order differential equation with variable coefficients is known, then its second solution can be obtained rather easily. Here we also demonstrate the method of variation of parameters to construct the solutions of nonhomogeneous second-order differential equations.

In Lecture 3 we provide some basic concepts which are required to construct power series solutions to differential equations with variable coefficients. Here through various examples we also explain ordinary, regular singular, and irregular singular points of a given differential equation. In Lecture 4 first we prove a theorem which provides sufficient conditions so that the solutions of second-order linear differential equations can be expressed as power series at an ordinary point, and then construct power series solutions of Airy, Hermite, and Chebyshev differential equations. These equations occupy a central position in mathematical physics, engineering, and approximation theory. In Lectures 5 and 6 we demonstrate the method of Frobenius to construct the power series solutions of second-order linear differential equations at a regular singular point. Here we prove a general result which provides three possible different forms of the power series solution. We illustrate this result through several examples, including Laguerre's equation, which arises in quantum mechanics. In Lecture 7 we study Legendre's differential equation, which arises in problems such as the flow of an ideal fluid past a sphere, the determination of the electric field due to a charged sphere, and the determination of the temperature distribution in a sphere given its surface temperature. Here we also develop the polynomial solution of the Legendre differential equation. In Lecture 8 we study polynomial solutions of the Chebyshev, Hermite, and Laguerre differential equations. In Lecture 9 we construct series solutions of Bessel's differential equation, which first appeared in the works of Euler and Bernoulli. Since many problems of mathematical physics reduce to the Bessel equation, we investigate it in somewhat more detail. In Lecture 10 we develop series solutions of the hypergeometric differential equation, which finds applications in several problems of mathematical physics, quantum mechanics, and fluid dynamics.

Mathematical problems describing real world situations often have solutions which are not even continuous. Thus, to analyze such problems we need to work in a set which is bigger than the set of continuous functions. In Lecture 11 we introduce the sets of piecewise continuous and piecewise smooth functions, which are quite adequate to deal with a wide variety of applied problems. Here we also define periodic functions, and introduce even and odd extensions. In Lectures 12 and 13 we introduce orthogonality of functions and show that the Legendre, Chebyshev, Hermite, and Laguerre polynomials and Bessel functions are orthogonal. Here we also prove some fundamental properties about the zeros of orthogonal polynomials.

In Lecture 14 we introduce boundary value problems for second-order ordinary differential equations and provide a necessary and sufficient condition for the existence and uniqueness of their solutions. In Lecture 15 we formulate some boundary value problems with engineering applications, and show that often solutions of these problems can be written in terms of Bessel functions. In Lecture 16 we introduce Green's functions of homogeneous boundary value problems and show that the solution of a given nonhomogeneous boundary value problem can be explicitly expressed in terms of Green's function of the corresponding homogeneous equation.

In Lecture 17 we discuss the regular perturbation technique which relates the unknown solution of a given initial value problem to the known solutions of the infinite initial value problems. In many practical problems one often meets cases where the methods of regular perturbations cannot be applied. In the literature such problems are known as singular perturbation problems. In Lecture 18 we explain the methodology of singular perturbation technique with the help of some examples.

If the coefficients of the homogeneous differential equation and/or of the boundary conditions depend on a parameter, then one of the pioneer problems of mathematical physics is to determine the values of the parameter (eigenvalues) for which nontrivial solutions (eigenfunctions) exist. In Lecture 19 we explain some of the essential ideas involved in this vast field, which is continuously growing.

In Lectures 20 and 21 we show that the sets of orthogonal polynomials and functions we have provided in earlier lectures can be used effectively as the basis in the expansions of general functions. This in particular leads to Fourier's cosine, sine, trigonometric, Legendre, Chebyshev, Hermite and Bessel series. In Lectures 22 and 23 we examine pointwise convergence, uniform convergence, and the convergence in the mean of the Fourier series of a given function. Here the importance of Bessel's inequality and Parseval's equality are also discussed. In Lecture 24 we use Fourier series expansions to find periodic particular solutions of nonhomogeneous differential equations, and solutions of nonhomogeneous self-adjoint differential equations satisfying homogeneous boundary conditions, which leads to the well-known Fredholm's alternative.

In Lecture 25 we introduce partial differential equations and explain several concepts through elementary examples. Here we also provide the most fundamental classification of second-order linear equations in two independent variables. In Lecture 26 we study simultaneous differential equations, which play an important role in the theory of partial differential equations. Then we consider quasilinear partial differential equations of the Lagrange type and show that such equations can be solved rather easily, provided we can find solutions of related simultaneous differential equations. Finally, we explain a general method to find solutions of nonlinear first-order partial differential equations which is due to Charpit. In Lecture 27 we show that like ordinary differential equations, partial differential equations with constant coefficients can be solved explicitly. We begin with homogeneous second-order differential equations involving only second-order terms, and then show how the operator method can be used to solve some particular nonhomogeneous differential equations. Then, we extend the method to general second and higher order partial differential equations. In Lecture 28 we show that coordinate transformations can be employed successfully to reduce second-order linear partial differential equations to some standard forms, which are known as canonical forms. These transformed equations sometimes can be solved rather easily. Here the concept of characteristic of second-order partial differential equations plays an important role.

The method of separation of variables involves a solution which breaks up into a product of functions each of which contains only one of the variables. This widely used method for finding solutions of linear homogeneous partial differential equations we explain through several simple examples in Lecture 29. In Lecture 30 we derive the one-dimensional heat equation and formulate initial-boundary value problems, which involve the heat equation, the initial condition, and homogeneous and nonhomogeneous boundary conditions. Then we use the method of separation of variables to find the Fourier series solutions to these problems. In Lecture 31 we construct the Fourier series solution of the heat equation with Robin's boundary conditions. In Lecture 32 we provide two different derivations of the one-dimensional wave equation, formulate an initial-boundary value problem, and find its Fourier series solution. In Lecture 33 we continue using the method of separation of variables to find Fourier series solutions to some other initial-boundary value problems related to one-dimensional wave equation. In Lecture 34 we give a derivation of the two-dimensional Laplace equation, formulate the Dirichlet problem on a rectangle, and find its Fourier series solution. In Lecture 35 we discuss the steady-state heat flow problem in a disk. For this, we consider the Laplace equation in polar coordinates and find its Fourier series solution. In Lecture 36 we use the method of separation of variables to find the temperature distribution of rectangular and circular plates in the transient state. Again using the method of separation of variables, in Lecture 37 we find vertical displacements of thin membranes occupying rectangular and circular regions. The three-dimensional Laplace equation occurs in problems such as gravitation, steady-state temperature, electrostatic potential, magnetostatics, fluid flow, and so on. In Lecture 38 we find the Fourier series solution of the Laplace equation in a three-dimensional box and in a circular cylinder. In Lecture 39 we use the method of separation of variables to find the Fourier series solutions of the Laplace equation in and outside a given sphere. Here, we also discuss briefly Poisson's integral formulas. In Lecture 40 we demonstrate how the method of separation of variables can be employed to solve nonhomogeneous problems.

The Fourier integral is a natural extension of Fourier trigonometric series in the sense that it represents a piecewise smooth function whose domain is semi-infinite or infinite. In Lecture 41 we develop the Fourier integral with an intuitive approach and then discuss Fourier cosine and sine integrals which are extensions of Fourier cosine and sine series, respectively. This leads to Fourier cosine and sine transform pairs. In Lecture 42 we introduce the complex Fourier integral and the Fourier transform pair and find the Fourier transform of the derivative of a function. Then, we state and prove the Fourier convolution theorem, which is an important result. In Lectures 43 and 44 we consider problems in infinite domains which can be effectively solved by finding the Fourier transform, or the Fourier sine or cosine transform of the unknown function. For such problems usually the method of separation of variables does not work because the Fourier series are not adequate to yield complete solutions. We illustrate the method by considering several examples, and obtain the famous Gauss–Weierstrass, d'Alembert's, and Poisson's integral formulas.

In Lecture 45 we introduce some basic concepts of Laplace transform theory, whereas in Lecture 46 we prove several theorems which facilitate the computation of Laplace transforms. The method of Laplace transforms has the advantage of directly giving the solutions of differential equations with given initial and boundary conditions without the necessity of first finding the general solution and then evaluating from it the arbitrary constants. Moreover, the ready table of Laplace transforms reduces the problem of solving differential equations to mere algebraic manipulations. In Lectures 47 and 48 we employ the Laplace transform technique to find solutions of ordinary and partial differential equations, respectively. Here we also develop the famous Duhamel's formula.

A given problem consisting of a partial differential equation in a domain with a set of initial and/or boundary conditions is said to be well-posed if it has a unique solution which is stable. In Lecture 49 we demonstrate that problems considered in earlier lectures are well-posed. Finally, in Lecture 50 we prove a few theorems which verify that the series or integral form of the solutions we have obtained in earlier lectures are actually the solutions of the problems considered.

Two types of exercises are included in the book, those which illustrate the general theory, and others designed to fill out text material. These exercises form an integral part of the book, and every reader is urged to attempt most, if not all of them. For the convenience of the reader we have provided answers or hints to almost all the exercises.

In writing a book of this nature no originality can be claimed, only a humble attempt has been made to present the subject as simply, clearly, and accurately as possible. It is earnestly hoped that *Ordinary and Partial Differential Equations* will serve an inquisitive reader as a starting point in this rich, vast, and ever-expanding field of knowledge.

We would like to express our appreciation to Professors M. Bohner, S.K. Sen, and P.J.Y. Wong for their suggestions and criticisms. We also want to thank Ms. Vaishali Damle at Springer New York for her support and cooperation.

Ravi P. Agarwal
Donal O'Regan

Contents

Lecture 1
Solvable Differential Equations

In this lecture we shall show that first-order linear differential equations with variable coefficients, second-order homogeneous differential equations with constant coefficients, and second-order Cauchy–Euler differential equations can be solved in terms of the known quantities.

First-order equations. Consider the differential equation (DE)

$$y' + p(x)y = q(x), \quad ' = \frac{d}{dx} \tag{1.1}$$

where the functions $p(x)$ and $q(x)$ are continuous in some interval J. The corresponding homogeneous equation

$$y' + p(x)y = 0 \tag{1.2}$$

obtained by taking $q(x) \equiv 0$ in (1.1) can be solved by separating the variables, i.e.,

$$\frac{1}{y}y' + p(x) = 0$$

and now integrating it, to obtain

$$\ln y(x) + \int^x p(t)dt = \ln c,$$

or

$$y(x) = c \exp\left(-\int^x p(t)dt\right). \tag{1.3}$$

In dividing (1.2) by y we have lost the solution $y(x) \equiv 0$, which is called the *trivial solution* (for a linear homogeneous DE $y(x) \equiv 0$ is always a solution). However, it is included in (1.3) with $c = 0$.

If $x_0 \in J$, then the function

$$y(x) = y_0 \exp\left(-\int_{x_0}^x p(t)dt\right) \tag{1.4}$$

clearly satisfies the DE (1.2) and passes through the point (x_0, y_0). Thus, this is the solution of the *initial value problem*: DE (1.2) together with the *initial condition*

$$y(x_0) = y_0. \tag{1.5}$$

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To find the solution of the DE (1.1) we shall use the *method of variation of parameters* due to Lagrange. In (1.3) we assume that c is a function of x, i.e.,

$$y(x) = c(x)\exp\left(-\int^x p(t)dt\right) \tag{1.6}$$

and search for $c(x)$ so that (1.6) becomes a solution of the DE (1.1). For this, setting (1.6) into (1.1), we find

$$\begin{aligned} c'(x)\exp\left(-\int^x p(t)dt\right) & -c(x)p(x)\exp\left(-\int^x p(t)dt\right) \\ & +c(x)p(x)\exp\left(-\int^x p(t)dt\right) = q(x), \end{aligned}$$

which is the same as

$$c'(x) = q(x)\exp\left(\int^x p(t)dt\right). \tag{1.7}$$

Integrating (1.7), we obtain the required function

$$c(x) = c_1 + \int^x q(t)\exp\left(\int^t p(s)ds\right)dt.$$

Now, substituting this $c(x)$ in (1.6), we find the solution of (1.1) as

$$y(x) = c_1\exp\left(-\int^x p(t)dt\right) + \int^x q(t)\exp\left(-\int_t^x p(s)ds\right)dt. \tag{1.8}$$

This solution $y(x)$ is of the form $c_1u(x)+v(x)$. It is to be noted that $c_1u(x)$ is the general solution of (1.2). Hence, the general solution of (1.1) is obtained by adding any particular solution of (1.1) to the general solution of (1.2).

From (1.8) the solution of the initial value problem (1.1), (1.5), where $x_0 \in J$, is easily obtained as

$$y(x) = y_0\exp\left(-\int_{x_0}^x p(t)dt\right) + \int_{x_0}^x q(t)\exp\left(-\int_t^x p(s)ds\right)dt. \tag{1.9}$$

Example 1.1. Consider the initial value problem

$$xy' - 4y + 2x^2 + 4 = 0, \quad x \neq 0, \quad y(1) = 1. \tag{1.10}$$

Since $x_0 = 1,\ y_0 = 1,\ p(x) = -4/x$ and $q(x) = -2x - 4/x$ from (1.9) the solution of (1.10) can be written as

$$\begin{aligned} y(x) &= \exp\left(\int_1^x \frac{4}{t}dt\right) + \int_1^x \left(-2t - \frac{4}{t}\right)\exp\left(\int_t^x \frac{4}{s}ds\right)dt \\ &= x^4 + \int_1^x \left(-2t - \frac{4}{t}\right)\frac{x^4}{t^4}dt \\ &= x^4 + x^4\left(\frac{1}{x^2} + \frac{1}{x^4} - 2\right) = -x^4 + x^2 + 1. \end{aligned}$$

Alternatively, instead of using (1.9) we can find the solution of (1.10) as follows: For the corresponding homogeneous DE $y' - (4/x)y = 0$ the general solution is cx^4, and a particular solution of the DE (1.10) is

$$\int^x \left(-2t - \frac{4}{t}\right) \exp\left(\int_t^x \frac{4}{s} ds\right) dt = x^2 + 1$$

and hence the general solution of the DE (1.10) is $y(x) = cx^4 + x^2 + 1$. Now, in order to satisfy the initial condition $y(1) = 1$, it is necessary that $1 = c+1+1$, or $c = -1$. The solution of (1.10) is, therefore, $y(x) = -x^4 + x^2 + 1$.

Second-order equations with constant coefficients. We shall find solutions of the second-order DE

$$y'' + ay' + by = 0, \tag{1.11}$$

where a and b are constants.

As a first step toward finding a solution to this DE we look back at the equation $y' + ay = 0$ (a is a constant) for which all solutions are constant multiples of e^{-ax}. Thus, for (1.11) also some form of exponential function would be a reasonable choice and would utilize the property that the differentiation of an exponential function e^{rx} always yields a constant multiplied by e^{rx}.

Thus, we try $y = e^{rx}$ and find the value(s) of r. We have

$$r^2 e^{rx} + are^{rx} + be^{rx} = 0,$$

or

$$(r^2 + ar + b)e^{rx} = 0,$$

or

$$r^2 + ar + b = 0. \tag{1.12}$$

Hence, e^{rx} is a solution of (1.11) if r is a solution of (1.12). Equation (1.12) is called the *characteristic polynomial* of (1.11). For the roots of (1.12) we have the following three cases:

1. Distinct real roots. If r_1 and r_2 are real and distinct roots of (1.12), then $e^{r_1 x}$ and $e^{r_2 x}$ are two solutions of (1.11), and its general solution can be written as

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

In the particular case when $r_1 = r,\ r_2 = -r$ (then the DE (1.11) is $y'' -$

$r^2y = 0$), we have

$$\begin{aligned} y(x) = c_1e^{rx} + c_2e^{-rx} &= \left(\frac{A+B}{2}\right)e^{rx} + \left(\frac{A-B}{2}\right)e^{-rx} \\ &= A\left(\frac{e^{rx}+e^{-rx}}{2}\right) + B\left(\frac{e^{rx}-e^{-rx}}{2}\right) \\ &= A\cosh rx + B\sinh rx. \end{aligned}$$

2. Repeated real roots. If $r_1 = r_2 = r$ is a repeated root of (1.12), then e^{rx} is a solution. To find the second solution, we let $y(x) = u(x)e^{rx}$ and substitute it in (1.11), to get

$$e^{rx}(u'' + 2ru' + r^2u) + ae^{rx}(u' + ru) + bue^{rx} = 0,$$

or

$$u'' + (2r+a)u' + (r^2 + ar + b)u \ = u'' + (2r+a)u' = 0.$$

Now since r is a repeated root of (1.12), it follows that $2r + a = 0$ and hence $u'' = 0$, i.e., $u(x) = c_1 + c_2x$. Thus,

$$y(x) = (c_1 + c_2x)e^{rx} = c_1e^{rx} + c_2xe^{rx}.$$

Hence, the second solution of (1.11) is xe^{rx}.

3. Complex conjugate roots. Let $r_1 = \mu + i\nu$ and $r_2 = \mu - i\nu$, where $i = \sqrt{-1}$, so that

$$e^{(\mu \pm i\nu)x} = e^{\mu x}(\cos\nu x \pm i\sin\nu x).$$

Since for the DE (1.11) the real part (i.e., $e^{\mu x}\cos\nu x$) and the imaginary part (i.e., $e^{\mu x}\sin\nu x$) both are solutions, the general solution of (1.11) can be written as

$$y(x) = c_1e^{\mu x}\cos\nu x + c_2e^{\mu x}\sin\nu x.$$

In the particular case when $r_1 = i\nu$ and $r_2 = -i\nu$ (then the DE (1.11) is $y'' + \nu^2y = 0$) we have $y(x) = c_1\cos\nu x + c_2\sin\nu x$.

Cauchy–Euler equations. For the *Cauchy–Euler equation*

$$t^2y'' + aty' + by = 0, \ \ t > 0 \ \ (t \text{ is the independent variable}), \ \ ' = \frac{d}{dt} \tag{1.13}$$

which occurs in studying the temperature distribution generated by a heat source such as the sun or a nuclear reactor, we assume $y(t) = t^m$ to obtain

$$t^2m(m-1)t^{m-2} + atmt^{m-1} + bt^m = 0,$$

or

$$m(m-1) + am + b = 0. \tag{1.14}$$

This is the characteristic equation for (1.13), and as earlier for (1.12) the nature of its roots determines the general solution:

Real, distinct roots $m_1 \neq m_2$: $y(t) = c_1 t^{m_1} + c_2 t^{m_2}$,

Real, repeated roots $m = m_1 = m_2$: $y(t) = c_1 t^m + c_2 (\ln t) t^m$,

Complex conjugate roots $m_1 = \mu + i\nu$, $m_2 = \mu - i\nu$: $y(t) = c_1 t^\mu \cos(\nu \ln t) + c_2 t^\mu \sin(\nu \ln t)$.

In the particular case

$$t^2 y'' + t y' - \lambda^2 y = 0, \quad t > 0, \quad \lambda > 0 \tag{1.15}$$

the characteristic equation is $m(m-1) + m - \lambda^2 = 0$, or $m^2 - \lambda^2 = 0$. Thus, the roots are $m = \pm\lambda$, and hence the general solution of (1.15) appears as

$$y(t) = c_1 t^\lambda + c_2 t^{-\lambda}. \tag{1.16}$$

Problems

1.1. (Principle of Superposition). If $y_1(x)$ and $y_2(x)$ are solutions of $y' + p(x)y = q_i(x)$, $i = 1, 2$ respectively, then show that $c_1 y_1(x) + c_2 y_2(x)$ is a solution of the DE $y' + p(x)y = c_1 q_1(x) + c_2 q_2(x)$, where c_1 and c_2 are constants.

1.2. Find general solutions of the following DEs:

(i) $y' - (\cot x)y = 2x \sin x$

(ii) $y' + y + x + x^2 + x^3 = 0$

(iii) $(y^2 - 1) + 2(x - y(1+y)^2)y' = 0$

(iv) $(1 + y^2) = (\tan^{-1} y - x)y'$.

1.3. Solve the following initial value problems:

(i) $y' + 2y = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}, \quad y(0) = 0$

(ii) $y' + p(x)y = 0, \quad y(0) = 1$ where $p(x) = \begin{cases} 2, & 0 \le x \le 1 \\ 1, & x > 1. \end{cases}$

1.4. Let $q(x)$ be continuous in $[0, \infty)$ and $\lim_{x\to\infty} q(x) = L$. For the DE $y' + ay = q(x)$ show that

(i) if $a > 0$, every solution approaches L/a as $x \to \infty$

(ii) if $a < 0$, there is one and only one solution which approaches L/a as $x \to \infty$.

1.5. Let $y(x)$ be the solution of the initial value problem (1.1), (1.5) in $[x_0, \infty)$, and let $z(x)$ be a continuously differentiable function in $[x_0, \infty)$ such that $z' + p(x)z \le q(x),\ z(x_0) \le y_0$. Show that $z(x) \le y(x)$ for all x in $[x_0, \infty)$. In particular, for the problem $y' + y = \cos x,\ y(0) = 1$ verify that $2e^{-x} - 1 \le y(x) \le 1,\ x \in [0, \infty)$.

1.6. Certain nonlinear first-order DEs can be reduced to linear equations by an appropriate change of variables. For example, this is always possible for the *Bernoulli equation:*

$$y' + p(x)y = q(x)y^n, \quad n \ne 0, 1.$$

Indeed this equation is equivalent to the DE

$$y^{-n}y' + p(x)y^{1-n} = q(x)$$

and now the substitution $v = y^{1-n}$ (used by Leibniz in 1696) leads to the first-order linear DE

$$\frac{1}{1-n}v' + p(x)v = q(x).$$

In particular, show that the general solution of the DE $xy' + y = x^2y^2,\ x \ne 0$ is $y(x) = (cx - x^2)^{-1},\ x \ne 0, c$.

1.7. Find general solutions of the following homogeneous DEs:

(i) $y'' + 7y' + 10y = 0$

(ii) $y'' - 8y' + 16y = 0$

(iii) $y'' + 2y' + 3y = 0$.

1.8. Show that if the real parts of the roots of (1.12) are negative, then $\lim_{x\to\infty} y(x) = 0$ for every solution $y(x)$ of (1.11).

1.9. Show that the solution of the initial value problem

$$y'' - 2(r + \beta)y' + r^2y = 0, \quad y(0) = 0, \quad y'(0) = 1$$

can be written as

$$y_\beta(x) = \frac{1}{2\sqrt{\beta(2r+\beta)}}\left[e^{[r+\beta+\sqrt{\beta(2r+\beta)}]x} - e^{[r+\beta-\sqrt{\beta(2r+\beta)}]x}\right].$$

Further, show that $\lim_{\beta\to 0} y_\beta(x) = xe^{rx}$.

1.10. The following fourth order DEs occur in applications as indicated:

(i) $y'''' - k^4y = 0$ (vibration of a beam)

(ii) $y'''' + 4k^4y = 0$ (beam on an elastic foundation)

(iii) $y'''' - 2k^2 y'' + k^4 y = 0$ (bending of an elastic plate),

where $k \neq 0$ is a constant. Find their general solutions.

Answers or Hints

1.1. Use the definition of a solution.

1.2. (i) $c \sin x + x^2 \sin x$ (ii) $ce^{-x} - x^3 + 2x^2 - 5x + 5$ (iii) $x(y-1)/(y+1) = y^2 + c$ (iv) $x = \tan^{-1} y - 1 + ce^{-\tan^{-1} y}$.

1.3. (i) $y(x) = \begin{cases} \frac{1}{2}(1 - e^{-2x}), & 0 \leq x \leq 1 \\ \frac{1}{2}(e^2 - 1)e^{-2x}, & x > 1 \end{cases}$ (ii) $y(x) = \begin{cases} e^{-2x}, & 0 \leq x \leq 1 \\ e^{-(x+1)}, & x > 1. \end{cases}$

1.4. (i) In $y(x) = y(x_0)e^{-a(x-x_0)} + [\int_{x_0}^x e^{at} q(t)dt]/e^{ax}$ take the limit $x \to \infty$ (ii) In $y(x) = e^{-ax}\left[y(x_0)e^{ax_0} + \int_{x_0}^\infty e^{at}q(t)dt - \int_x^\infty e^{at}q(t)dt\right]$ choose $y(x_0)$ so that $y(x_0)e^{ax_0} + \int_{x_0}^\infty e^{at}q(t)dt = 0$ ($\lim_{x\to\infty} q(x) = L$). Now in $y(x) = -[\int_x^\infty e^{at}q(t)dt]/e^{ax}$ take the limit $x \to \infty$.

1.5. There exists a continuous function $r(x) \geq 0$ such that $z' + p(x)z = q(x) - r(x),\ z(x_0) \leq y_0$. Thus, for the function $\phi(x) = y(x) - z(x),\ \phi' + p(x)\phi = r(x) \geq 0,\ \phi(x_0) = y_0 - z(x_0) \geq 0$.

1.6. Using the substitution $v = y^{-1}$ the given equation reduces to $-xv' + v = x^2$.

1.7. (i) $c_1 e^{-2x} + c_2 e^{-5x}$ (ii) $(c_1 + c_2 x)e^{4x}$ (iii) $c_1 e^{-x} \cos\sqrt{2}x + c_2 e^{-x} \times \sin\sqrt{2}x$.

1.8. Use explicit forms of the solution.

1.9. Note that $\sqrt{\beta(\beta + 2r)} \to 0$ as $\beta \to 0$.

1.10. (i) $c_1 e^{kx} + c_2 e^{-kx} + c_3 \cos kx + c_4 \sin kx$ (ii) $e^{kx}(c_1 \cos kx + c_2 \sin kx) + e^{-kx}(c_3 \cos kx + c_4 \sin kx)$ (iii) $e^{kx}(c_1 + c_2 x) + e^{-kx}(c_3 + c_4 x)$.

Lecture 2
Second-Order Differential Equations

Generally, second-order differential equations with variable coefficients cannot be solved in terms of the known functions. In this lecture we shall show that if one solution of the homogeneous equation is known, then its second solution can be obtained rather easily. Further, by employing the method of variation of parameters, the general solution of the nonhomogeneous equation can be constructed provided two solutions of the corresponding homogeneous equation are known.

Homogeneous equations. For the homogeneous linear DE of second-order with variable coefficients

$$y'' + p_1(x)y' + p_2(x)y = 0, \tag{2.1}$$

where $p_1(x)$ and $p_2(x)$ are continuous in J, there does not exist any method to solve it. However, the following results are well-known.

Theorem 2.1. There exist exactly two solutions $y_1(x)$ and $y_2(x)$ of (2.1) which are linearly independent (essentially different) in J, i.e., there does not exist a constant c such that $y_1(x) = cy_2(x)$ for all $x \in J$.

Theorem 2.2. Two solutions $y_1(x)$ and $y_2(x)$ of (2.1) are linearly independent in J if and only if their *Wronskian* defined by

$$W(x) = W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \tag{2.2}$$

is different from zero for some $x = x_0$ in J.

Theorem 2.3. For the Wronskian defined in (2.2) the following Abel's identity holds:

$$W(x) = W(x_0)\exp\left(-\int_{x_0}^{x} p_1(t)dt\right), \quad x_0 \in J. \tag{2.3}$$

Thus, if Wronskian is zero at some $x_0 \in J$, then it is zero for all $x \in J$.

Theorem 2.4. If $y_1(x)$ and $y_2(x)$ are solutions of (2.1) and c_1 and c_2 are arbitrary constants, then $c_1y_1(x) + c_2y_2(x)$ is also a solution of (2.1).

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